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Some Global and Local Aspects of Bigravity

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Abstract We review some global and local aspects of bigravity theories concerning the causal structure of two interacting metrics. We provide some general settings where global hyperbolicity is lost even if both metrics are independently globally hyperbolic. Finally we say some words about the spectrum of these theories and their relation to the cosmological constant problem.

1 Introduction

The current observed acceleration at cosmological distances in the universe supposes one of the most important puzzles of current physics. Its most popular solution is to invoke some exotic kind of matter (dubbed *dark energy*) which gives rise to a positive cosmological constant that drives this acceleration. An alternative proposal is to modify Einstein's theory of gravity in order to account for the acceleration in a natural way. This philosophy opens the door to a whole bunch of new theories which are phenomenologically viable and still explain the acceleration of the universe without the need of any exotic kind of matter.[1] Among them, we may distinguish between those theories based on a low-energy description of a more fundamental theory (such as those invoking higher dimensional dynamics, e.g. [1]) and those which are *ad hoc* (as MOND at its current status [2, 3]). In this work we will deal with a possibility which somehow lies between both realms, namely *massive gravity* from *bigravity*. Indeed we would like to put forward the idea of a massive graviton to be responsible for the weaker behavior of gravity at large distances, and we may try to find mechanisms which give rise to this mass. An interesting possibility is that the mass term

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[1] Somehow the exotic features are moved to the gravity sector.

(which violates the gauge invariance of the theory) is produced by the effective quantum interaction of the graviton with the background without a cosmological constant, as happens for spin one particles at finite temperature. We leave this possibility for further research [4] and concentrate in classical modification of the graviton propagator through it interaction with classical fields. One of the features of massive gravity is that even if at the linear order a mass term can be constructed for the perturbation field, at the non-linear level such a covariant term does not exist unless we add new fields to the theory. These new fields may be non-dynamical (we can add a background metric as we do at the linearized level, see [5] and references therein) or dynamical and may produce the mass term due to the spontaneous breaking of the symmetry. Some models in this direction involving a ghost condensation have been popular recently [6, 7], but here we are going to pursue another proposal. Furthermore, the linear theory of massive gravity is full of inconsistencies with experiment which may be cured at the non-linear level [8–19] and bigravity can be considered as an easy setup which may shed some light on this topic.

In this paper we will consider the possibility of a universe with two interacting metrics, each of them coupled to a sector of matter. Some work in this direction, as long as more motivation (in particular some fundamental settings where this possibility appears), can be found in [20–23] and references therein. We will first review some issues about the global structures of some solutions. This is related to the fact that two causal structures coexist in the theory, and many concepts of ordinary Lorentzian manifolds may be (almost trivially) generalize to this setup. Issues such as global hyperbolicity or geodesic completeness refer now not to the whole manifold but to a particular metric, and thus their global definition for both metrics simultaneously may lead to surprises such as non-globally hyperbolic structures built out of two globally hyperbolic metrics. After studying some features of solutions to the field equations, we will show some general settings where the tension between global hyperbolicity and geodesic completeness for non-coinciding horizons can be noticed. It is also interesting to note that among the solutions of bigravity we recover the standard ones for GR, which means that some of the problems of linearized gravity disappear in this setup.

In the remaining of the paper, we will consider perturbations to some solutions and find the spectrum which will consist of two massive gravitons propagating in these backgrounds. The appearance of generic mass terms different from the usual Fierz–Pauli one (even not covariant), will motivate the study of the most general mass term for massive gravity in de Sitter space. We will determine the conditions for ghost and tachyon free solutions in these backgrounds and find a wider family of possibilities that those of the Minkowski case [24] and the FP in de Sitter [25]. Finally, we will make some comments concerning the possible relations of this approach to the cosmological constant problem.

2 Global Aspects of f–g Bigravity

In this section we will review some material that appeared in [20] where references to earlier work can be found. Let us define the action of the bigravity theory as[2]

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{-R_g}{2\kappa_g} + L_g\right) + \int \mathrm{d}^4x\sqrt{-f}\left(\frac{-R_f}{2\kappa_f} + L_f\right) + S_{\mathrm{int}}[f, g]. \tag{1}$$

[2]We will adopt the *weak coupled worlds* approach where there are two kinds of matter, each of which coupled to one metric [23].

Here L_f and L_g denote generic matter Lagrangians coupled to the metrics f and g respectively. In solving the field equations, we shall restrict attention to the case where there is only a vacuum energy term in each matter sector $L_f = -\rho_f$, $L_g = -\rho_g$. There is much freedom in the choice of the interaction term in (1). For definiteness, we shall for the moment consider the form [26]

$$S_{\text{int}} = -\frac{\zeta}{4}\int \mathrm{d}^4x(-g)^u(-f)^v(f^{\mu\nu} - g^{\mu\nu})(f^{\sigma\tau} - g^{\sigma\tau})(g_{\mu\sigma}g_{\mu\tau} - g_{\mu\nu}g_{\sigma\tau}), \tag{2}$$

with

$$u + v = \frac{1}{2}. \tag{3}$$

This reduces to the Pauli–Fierz one in the linear regime, when g is taken to be the Minkowski metric (but many other choices could be made with this same property [23]). The equations of motion, derived from action (1), read

$$G^f_{\mu\nu} = \kappa_f T^f_{\mu\nu}, \qquad G^g_{\mu\nu} = \kappa_g T^g_{\mu\nu}, \tag{4}$$

where $G_{\mu\nu}$ denotes the Einstein tensor for each metric and the form of $T_{\mu\nu}$ can be found in [20].

Let us now concentrate on static spherically symmetric solutions. By suitable choice of coordinates, the most general static and spherically symmetric ansatz for bigravity can be cast into the form

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = J\mathrm{d}t^2 - K\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{5}$$

$$f_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = C\mathrm{d}t^2 - 2D\mathrm{d}t\mathrm{d}r - A\mathrm{d}r^2 - B(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{6}$$

where the metric coefficients are functions of r. As noted in Ref. [27], with the present ansatz we have

$$DG^f_{tt} + CG^f_{tr} = 0. \tag{7}$$

Hence, from the equations of motion, the following combination of components of the energy momentum tensor T^f must also vanish

$$DT^f_{tt} + CT^f_{tr} = -\frac{\zeta DJ}{2B}\left(\frac{JKr^4}{\Delta B^2}\right)^u(3B - 2r^2) = 0. \tag{8}$$

Here, we have introduced $\Delta = AC + D^2$ [27]. There are two different ways of satisfying this equality. Either

$$B = \frac{2}{3}r^2, \tag{9}$$

or

$$D = 0. \tag{10}$$

Solutions obeying the first condition (9) are called type I, and they were first discussed in [28]. The remaining ones are called type II and, from (10), they are necessarily diagonal.

In the case of type I solutions, the general solution can be written as

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = (1-q)\mathrm{d}t^2 - (1-q)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{11}$$

$$f_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \frac{2}{3\beta}(1-p)\mathrm{d}t^2 - 2D\mathrm{d}t\mathrm{d}r - A\mathrm{d}r^2 - \frac{2}{3}r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{12}$$

where

$$A = \frac{2}{3\beta}(1-q)^{-2}(p+\beta-q-\beta q), \tag{13}$$

$$D^2 = \left(\frac{2}{3\beta}\right)^2(1-q)^{-2}(p-q)(p+\beta-1-\beta q). \tag{14}$$

Here $\beta > 0$ is an arbitrary constant, and the potentials p and q are functions of r. Substituting this ansatz into the expressions for the energy-momentum tensors, they take the form of cosmological terms: $T^f_{\mu\nu} = (\Lambda_f/\kappa_f)f_{\mu\nu}$, and $T^g_{\mu\nu} = (\Lambda_g/\kappa_g)g_{\mu\nu}$, where

$$\frac{\Lambda_f}{\kappa_f} = \frac{\zeta}{4}\left(\frac{3}{2}\right)^{4u}\beta^u\{3v + 9\beta(1-v)\} + \rho_f, \tag{15}$$

$$\frac{\Lambda_g}{\kappa_g} = \frac{\zeta}{4}\left(\frac{2}{3}\right)^{4v}\beta^{-v}\{3u - 9\beta(1+u)\} + \rho_g. \tag{16}$$

Thus, these solutions correspond to a family of two Schwarzschild–(A)de Sitter metrics. Notice, however, that the cosmological constants are not determined by the vacuum energies ρ_f and ρ_g and the parameters in the Lagrangian, but they also depend on an arbitrary integration constant β. This is reminiscent of unimodular gravity where the vacuum energy does not determine the trace of the Ricci scalar (see [29, 30] and references therein).

Notice that taking the massless limit in the non-linear theory would amount to set $\zeta = 0$. From the previous expressions we see that this limit produces the standard solutions of GR for both metrics and thus the vDVZ discontinuity disappears in this set up (see also [12–19]).

Type II solutions are defined as those which do not satisfy (9), and therefore must be diagonal. Although the general type II solution is not known, some progress can be made by further assuming that one of the metrics, say g, is a solution of

$$G^g_{\mu\nu} = \Lambda g_{\mu\nu}, \tag{17}$$

for some constant Λ. In [20], we showed that the only type II solutions in this subclass are such that

$$f_{\mu\nu} = \gamma g_{\mu\nu}, \tag{18}$$

where γ is a constant (to be determined below).

Substituting (18) into the energy-momentum tensors yields a diagonal energy-momentum tensor for both the g and the f metrics, with respective cosmological constants given by

$$\Lambda_g = -6\kappa_g\zeta\gamma^{4v}(-1+\gamma)(-1-2u+2\gamma u)/(2\gamma^2) + \kappa_g\rho_g, \tag{19}$$

$$\Lambda_f = -6\kappa_f\zeta\gamma^{-4u}(-1+\gamma)(1-2v+2\gamma v)/(2\gamma^2) + \kappa_f\rho_f. \tag{20}$$

Equation (18) implies that the Einstein tensors for both metrics are identical, and thus

$$\Lambda_g = \gamma \Lambda_f. \tag{21}$$

This is an algebraic equation which determines γ in terms of the parameters in the Lagrangian.

Solutions satisfying (17) and (18) exist for arbitrary interaction between both metrics (not necessarily of the form (2)). The reason is that by assuming the ansatz where $f_{\mu\nu} = \gamma g_{\mu\nu}$, we necessarily have

$$T^g_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S_{\rm int}[f, g]}{\delta g^{\mu\nu}} = \Lambda_g(\gamma)\, g_{\mu\nu},$$

with constant Λ_g, and similarly for T^f. All that will change from one theory to the other is the form of the equation $\Lambda_g(\gamma) = \gamma \Lambda_f(\gamma)$ which determines γ [21]. We will have more to say about general bigravities later on, when we study their possible application to the cosmological constant problem. A similar argument can be made for multigravity theories, where one can always find solutions where all metrics are proportional to each other (see e.g. [31]).

As in the case of type I, both f and g solve Einstein's equations with a cosmological term. In this sense, there is no sign of a vDVZ discontinuity either, and one finds the usual Schwarzschild–(A)dS metrics, without modification. Unlike type II solutions, here the two metrics share the same light cones and therefore their causal structure does not pose any novelties.

2.1 Global Causal Structure

As stated in the introduction, the coexistence of two metric structures in a manifold raises many questions about their compatibility. Even if the manifold is orientable and admits a Lorentzian metric, the existence of another Lorentzian metric with completely different causal behavior is possible in principle (e.g., the existence of closed time-like curves (CTC), geodesically completeness or global hyperbolicity). Every notion on causality must thus refer exclusively to a single metric and the term of interaction is the only constraint to the existence of the different pathologies we may imagine.[3]

Let us first consider the issues of causal compatibility, maximal extensions and geodesic completeness. To study them we will make a conformal compactification of one of the metrics and see how do the null geodesics for the other metric look like in this (finite) diagram. Notice that the existence of a common group of symmetry $SO(3)$, allows us to focus on radial geodesics. Besides, the conformal compactification is also useful to extend the geodesics of the compactified metric which reached the boundary in a finite proper time to find the maximal extension of one of the metrics. However, if the companion metric is already geodesically complete, for this metric this amount to the addition of a completely new piece of spacetime which is not reachable by its own curves. Even if this may sound quite exotic,

[3] In principle, the existence of two non-vanishing vector fields (each of them corresponding to one of the metrics) related in a given way which depends on the relation between the metrics, would restrict the topology of the manifold but we will not consider this issue here. Also, whenever the metric is not *time-orientable* it is customary to use the double-covering of the manifold (cf. [32]). For two metrics it is conceivable to have closed curves which change the time orientation of both metrics, of a single metric or of none of them, and thus one may need a four-covering space of the manifold whose definition is the trivial generalization of the double-covering.

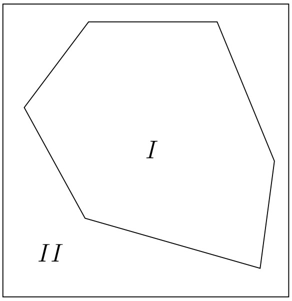

Fig. 1 Conformal compactification of two conformally related metrics. Notice that even if the light-cones structure is the same, the conformal boundary can be completely different

it is worth noting that translated to the ordinary language of GR this simply corresponds to the existence of Cauchy horizons for what we thought to be a globally hyperbolic manifold, due to the interaction of the metric with another causal structure.

First we will say a few words about conformally related metrics. If the metrics can be related by a general conformal factor, $f_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$ even if the null cones will be the same at every point of the manifold, the global issues can differ.[4] In particular, if the conformal factor is not asymptotically a constant so that both metric have the same asymptotic behavior, i.e., if it diverges or cancels, a causal curve for one of the metrics can reach the boundary in a finite time whereas for the other metric the boundary is not reachable in a finite proper time through any causal curve. In such cases, just one of the metrics should be extended through the boundary and no conformal factor exists that can put both infinities at the same place in the conformal diagram. Thus, the general diagram would be something similar to Fig. 1 where by region I we mean the region where both metrics are defined at first and by region *II* we mean the maximal extension of one of the metrics. In fact, if we want *II* to be a bigravity system, we must add to the extended metric a new companion for which the boundary between I and *II* is at infinite proper distance and so that they constitute a solution of the e.o.m. without any relation to the previous non-extended metric in I which was already geodesically complete (as we said, this is similar to what happens whenever Cauchy horizons are present in GR). Besides, it is clear that a global Cauchy surface can be built provided that the metric in the new region still is conformal to the extended metric. To find a solution, this process may require a new metric which does not share the conformal boundary of the extended metric, and thus new patches should be necessary to describe a geodesically complete solution, and global hyperbolicity may be lost in this process (the first extended metric can now develop a Cauchy horizon for this extension. We will have more to say about this issue for non-conformally related metrics). Clearly, when both metrics are proportional, the conformal diagram and the maximal extensions are exactly the same for both of them.

As far as the case of non-proportional metric is concerned, we will study the features of a concrete type I solution (see also [20]). In particular, we can choose parameters in (15–16) so that $\Lambda_g = 0$ and $\Lambda_f > 0$. Then there is a type I solution where g is Minkowski and f is de Sitter. The corresponding potentials in (11–12) are given by

$$p = \frac{2\Lambda_f}{9} r^2 \equiv H^2 r^2, \qquad q = 0. \tag{22}$$

[4] Remember, for instance, that given a metric with singularities and satisfying certain plausible physical condition, a conformal factor always exists that set the singularities at an infinite distance [32]. However, for two metrics, this remark does not apply in general.

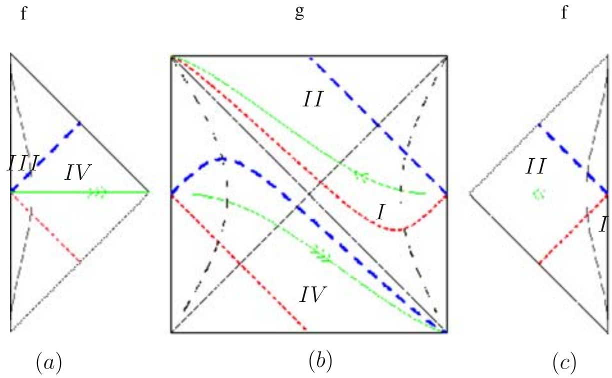

Fig. 2 Diagram showing the extension proposed in the text for the de Sitter(f)/Minkowski(g) solution. The *dashed curly vertical line* of the *left diagram* represents a sphere of constant radial coordinate r. The *solid curly vertical line* of the *right diagram* represents the de Sitter horizon $r = r_H$ plotted in the Minkowski spacetime. We also plotted three radial geodesics of Minkowski spacetime emanating from the origin $r = 0$ at $t = 0$: the *thick dashed* (*blue*) curve is a future-directed radial null ray from the origin (notice it is also a null geodesic ($V =$ constant) of the de Sitter spacetime), the *thin solid* (*green*) curve with two arrows is a $t = 0$ radial geodesic, the *thin dashed* (*red*) curve is a past-directed null ray from the origin. The last two curves are radial geodesics of Minkowski spacetime but not of de Sitter spacetime. The whole of the Minkowski spacetime is mapped onto the half of the de Sitter diagram verifying $V > 0$. By using a second Minkowski space, we extend the de Sitter/Minkowki system of regions *I* and *II* to a geodesically complete set-up

Note that each of the spacetimes, characterized respectively by the metrics (11) and (12) with the above defined potentials, has a maximal extension which is geodesically complete (trivial in the case of Minkowski). However, combining both together will be non-trivial because the static coordinates (t, r) (where we also include implicitly the angular part) cover the whole of Minkowski space, but not the whole of de Sitter. Hence, the conformal diagram for the extended de Sitter space accommodates all points for which the metric g is defined, but the converse is not true. To illustrate the causal structure, let us represent the light-cones of metric g in the conformal diagram of f. The details about this process can be found in [20]. The result is that in the usual Kruskal-coordinates for de Sitter (U, V), the radial null geodesics for Minkowski

$$t = \epsilon r + k, \tag{23}$$

where $\epsilon = \pm 1$ refers to future and past directed null geodesics, are expressed as

$$U = \left(\frac{Hr - 1}{Hr + 1}\right) e^{-Hk} e^{-H(\epsilon-1)r}, \qquad V = e^{Hk} e^{H(\epsilon-1)r}. \tag{24}$$

The form of one of these light-cones can be seen in regions *I* and *II* of Fig. 2. There we see that even if at $r = 0$ the light cone coincide, the future directed geodesic begins to bend till it progress towards the future for the de Sitter metric! Nevertheless, it never enters again the light-cone for de Sitter, which prevents from the existence of CTC made up with both metrics. Furthermore, the coordinates (r, t) cover the full Minkowski space corresponding to the metric g, but only half of the conformal diagram for the extended de Sitter metric, corresponding to $V > 0$ (see Figs. 2 and 3). This portion is by itself globally hyperbolic, since the $t = k$ surfaces are Cauchy surfaces for all geodesics of both metrics in this region. However, the region $V > 0$ is not geodesically complete, since the null geodesics $U =$ const of de Sitter reach $V = 0$ at finite affine parameter. To obtain a geodesically complete spacetime, we can match the solution in the upper half of the conformal diagram with a solution in the lower half of the diagram. For this purpose we introduce a *second* Minkowski space, with metric g', which will be covered with coordinates r' and t'. The usual change of variables between Kruskal (U, V) and the static coordinates (r, t) with the substitutions $t \to -t'$, $U \to -U$, $V \to -V$, maps the full range of the coordinates r', t' into the lower half of the de Sitter conformal diagram, below the diagonal $V = 0$. The full diagram, represented by the four regions in Fig. 2, is now geodesically complete. In doing such an extension, we mean we are gluing together one Minkowski spacetime to the other along the past infinity of the $r = r_H$ sphere of the former to the future infinity of the $r = r_H$ sphere of the latter. These infinities do not belong to the Minkowski spacetimes, but to their boundaries, while they are located in the interior of the de Sitter spacetime. This provides indeed a perfectly fine geometric maximal extension, where all geodesics are complete.

We should add, however, that a maximal extension is usually required to satisfy the equations of motion. The bigravity equations of motion are certainly satisfied everywhere in regions I, II, III and IV of Fig. 2, but it is unclear in which sense they are satisfied along the diagonal $V = 0$. The problem is precisely that we are joining two Minkowski spacetimes ((a) and (c) of Fig. 2) at a locus which lies at their conformal boundary. It is conceivable that promoting our maximal extension to a solution of the equations of motion might necessitate additional input, such as the inclusion of some source at the time-like infinity of Minkowski.

Furthermore, as we shall see below, the extensions are not unique. This may seem surprising at first sight, but as we have already commented a similar ambiguity is present in usual General Relativity when a metric must be continued beyond a Cauchy horizon.

Concerning the global hyperbolicity, the extended diagram, Fig. 2, is not globally hyperbolic. Indeed, the $t = k$ surfaces of the region $V > 0$ are no longer Cauchy surfaces for the whole spacetime, since they do not intersect causal geodesics in the lower half of the diagram. A surface which intersects all causal geodesics should cut through both regions, $V > 0$ as well as $V < 0$. One such surface is, for instance, the horizontal line $U = V$. The problem is that, as can be seen in Fig. 2, there are geodesics which intersect this surface twice (such as the past directed null rays from $r = t = 0$). A formal proof that the maximally extended diagram of Fig. 2 is not globally hyperbolic runs as follows. Let us restrict attention to radial geodesics. A Cauchy surface must intersect all causal geodesics once and only once. Let us assume that such a surface Σ exists. In particular, Σ must intersect the null geodesic $V = 0$ of de Sitter space. By continuity, it will also intersect the null geodesics $V = \text{const.}$, in the range $-\delta < V < \delta$, where δ is an arbitrarily small positive number. Let us now consider the null geodesic of Minkowski space, parametrized by r in (24), and let us choose the constant $k < H^{-1} \ln \delta$. It is clear that the incoming radial geodesic (with $\epsilon = -1$) will start at the upper left corner of the de Sitter diagram (at $r \to \infty$), and work its way down towards the right boundary of the diagram (at $r = 0$), while V will always remain in the interval $0 < V < \delta$). Hence, the incoming null geodesic must intersect Σ at least once before it reaches $r = 0$. At $r = 0$ it bounces and becomes the outgoing null geodesic $V = e^{Hk} < \delta$, which will intersect Σ once more before it reaches null future infinity. Hence, there are geodesics of Minkowski which intersect Σ twice, which simply means that this is not a good Cauchy surface for all geodesics in the extended diagram. In the next section we will prove some general theorems from which this proof is a corollary.

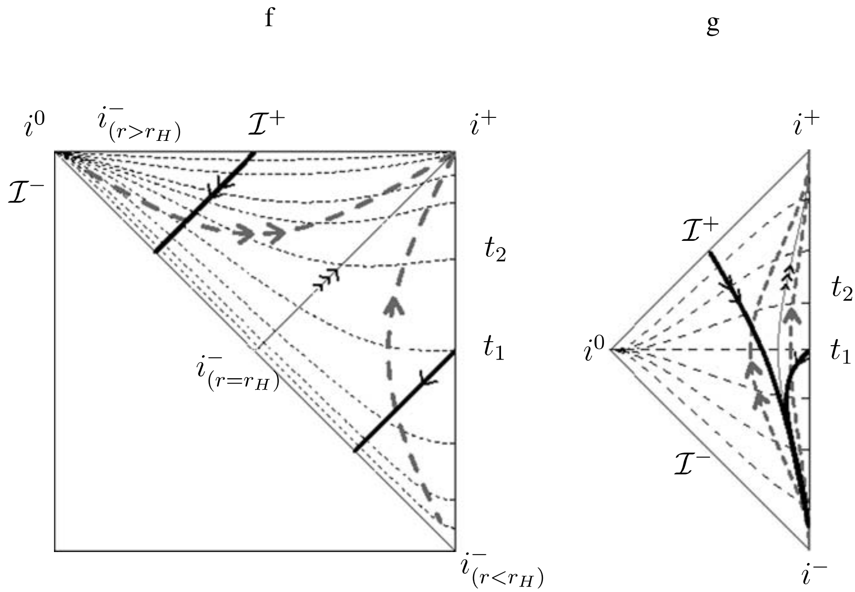

Fig. 3 Causal diagram for de Sitter with Minkowski, for $\beta = 1$. The *left diagram* is for de Sitter with horizon radius r_H, while the *right diagram* is for Minkowski. The *dashed thin lines* (*with no arrows*) are $t =$ constant lines. The *dashed thick line with one* (resp. *two*) *arrow* is an $r =$ constant curve, with $r < r_H$ (resp. $r > r_H$). The *thin solid line with three arrows* represents the trajectory of an observer sitting at constant radius $r = r_H$ in Minkowski spacetime. The *thick solid lines with arrows* are past directed null geodesics of de Sitter space time $U =$ constant curves. The mapping of the infinities (null, spacelike, timelike) of Minkowski spacetimes ($i^{\pm,0}, \mathcal{I}^{\pm}$) has been indicated on the de Sitter diagram. One of the striking features of those diagrams, is that the past time-like infinity of Minkowski is split between the *upper left corner* (for $r > r_H$), the *lower right corner* (for $r < r_H$) and the diagonal ($r = r_H$) of the de Sitter spacetime

We may study how is the conformal structure of Minkowski mapped into the boundary of the de Sitter diagram.[5] The future timelike infinity i^+ of Minkowski is mapped into the upper right corner of the de Sitter diagram, the future null infinity $\mathcal{I}^+$ of Minkowski is mapped into the future null infinity of de Sitter (which is spacelike), the spacelike infinity i^0 and null past infinity $\mathcal{I}^-$ of Minkowski are both mapped to the upper left corner of the de Sitter diagram (see Fig. 3). The situation is more complicated for the past timelike infinity i^- of Minkowski. The latter is split into three pieces: a particle moving back in time along a $r =$ const geodesic of Minkowski spacetime would either go to the upper left corner of the de Sitter diagram if $r > r_H$, to the lower right corner if $r < r_H$, or to the $U = 0$, $V = 0$ central point if $r = r_H$. However, a given timelike trajectory in Minkowski, stemming from the infinite past ($t = -\infty$, $r = r_H$) can emanate in the de Sitter diagram from any point along the diagonal $V = 0$. The latter diagonal is then representing the whole of the past $r = r_H$ infinity of Minkowski. This can be better seen, plotting the null geodesics of de Sitter into a conformal diagram for Minkowski.

More examples of these kind of extensions can be found in [20]. In particular, when the two metrics have non-coinciding horizons, both of them must be supplemented with new

[5] Remember that when we consider a conformal compactification of a given manifold, the topology of the boundary is attributed with respect to the characteristics of the curves that arrive to this boundary. This is why we can have different topologies for the boundary of a given patch for different metrics.

patches of spacetime in order to find maximal extensions. Thus we arrive to a kind of stair which only ends when one of the metrics has no horizons (cf. [20]).

Two interesting phenomena happen here. First of all, we may wonder about the Cherenkov radiation emitted by the particles which travel faster than light with respect to one of the metrics, as they couple to the metric whose light-cones may be more open. Indeed, this is analogous to the situation found in condense matter where two effective metrics coexist in the solid: the background metric and the metric induced by the interaction with the solid. Besides, the appearance of Cauchy horizons is usually a sign of instability of the system due to the accumulation of perturbations close to the horizon (see [33–36]). Thus, we expect all the metrics with non-coinciding horizons to be classically unstable.

2.2 Closed Time-Like Curves

An interesting question regarding bigravity solutions is whether we can construct closed time-like curves by sewing together curves which are time-like just for one of the metrics. For conformally related metrics it is clear that this can not be done as the null cones are the same in both cases.

It is quite interesting to note that for all the spherically symmetric solutions shown in this section, and in spite of the strong differences in the light-cone structure, these CTC do not exist. To prove this we showed that even if the future (past) directed time-likes curves for one of the metrics can go backwards (forward) in time for the other one, it never enters the past (future) light-cone, and thus no CTC can be constructed for any type I solution. More details about this behavior can be found in [20].

3 Global Hyperbolicity vs. Geodesic Completeness

In this section we will study some general settings where global hyperbolicity for geodesically complete and globally hyperbolic manifolds with respect to two metrics (not necessarily solutions of the equations of motion) is lost. These settings can be found in some situations where one of the metrics has a horizon which is not shared by the other one or when both metrics share a horizon but they are of different type. In particular, the proof of the lost of global hyperbolicity of the setting in the previous section follows as a corollary, as well as the proofs presented in [20] for other situations. The main idea of these settings can be acquired from Fig. 4. We will use the notation and conventions of [32]. A subindex f or g will indicate that the concept refers to the metric f or g respectively.

Lemma 1 *Let $\mathcal{M}$ be a time orientable manifold endowed with two globally hyperbolic metrics f and g. Let us suppose that there exist a point $p \in \overline{\mathcal{M}}$ through which the manifold can be extended for the metric g through the past (future). Consider a family λ_p of future (past)*

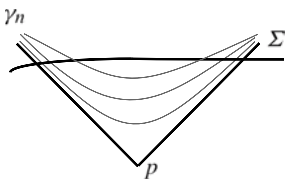

Fig. 4 This figure gives a general idea of the settings in this section. Σ is a Cauchy surface for the metric g for which the lightcone from p is drawn. $\{\gamma_n\}$ is a series of spacelike curves for g which converge to a curve in T_p^+ and to a timelike curve for f

directed non-spacelike curves for the g metric stemming from p. If there is a Cauchy surface Σ which intersects $J^+(p)$ for g and there is a non-causal curve for g, $\gamma \in \mathrm{int}(D^+(\lambda_p, \mathcal{M}))$ such that γ is non-compact and without boundary in $\mathrm{int}(D^+(\lambda_p, \mathcal{M}) \cap D_g^-(\Sigma))$ but it is compact in $D^+(\lambda_p, \mathcal{M}) \cap D_g^-(\Sigma)$, if $\gamma \cap D_g^-(\Sigma)$ is timelike for the companion f metric, Σ will not be a Cauchy surface for f.

Proof As γ is timelike for f which is globally hyperbolic, it can not be a self intersecting curve. Thus, being compact and not-self intersecting, γ will have two boundary points q_1 and q_2 in $D^+(\lambda_p, \mathcal{M}) \cap D_g^-(\Sigma)$ or will be a S^1. As $\gamma \cap \dot{D}^+(\lambda_p, \mathcal{M}) = \emptyset$ and is non-compact and without boundary in $\mathrm{int}(D^+(\lambda_p, \mathcal{M}) \cap D_g^-(\Sigma))$, these points can only by in Σ and in the case of the S^1 it will contain a point in Σ. Thus, the curve intersects the Cauchy surface at least twice (for the S^1, it will intersect it a infinite number of times). But as γ is timelike for f, Σ will not be a Cauchy surface for f as timelike curve will intersect it more than once. □

Once we have proved the previous lemma the issue to show the non-compatibility between global hyperbolicity and geodesic completeness is to build the curve γ. Let us consider the future null cone for the metric g at a point p of the boundary of a manifold $\mathcal{M}$, i.e., $p \in \overline{\mathcal{M}}$. If $\overline{\mathcal{M}}$ is b-complete, the light rays in the null cone can be approached by both connected timelike and connected spacelike curves in all the disconnected parts in which $\overline{\mathcal{M}}$ is divided by the cone. In the regions in the future and past of p, the lightcone T_p can be approached by spacelike curves $\{\gamma_n\}$ stemming from the same points as T_p in $\overline{\mathcal{M}}$, whereas for the regions connected to p by timelike curves, these curves are timelike. Suppose that these curves belong to $\mathcal{M}$. This means that they must converge to a curve γ_g in $\overline{\mathcal{M}}_f$ and similarly to a curve γ_f in $\overline{\mathcal{M}}_g$.[6] For the g metric, this curve is composed of two future directed null curves stemming from p, and thus every Cauchy surface will have to intersect both curves in $J^+(p)$ or $J^-(p)$ or at p. Let us suppose that it intersects $J^+(p)$. As the surface Σ must be spacelike for both f and g, it will also intersect twice the curves γ_n for $n \geq m$. However, if $\gamma_g \cap \mathcal{M}$ is timelike so will be the curves γ_n for $n \geq p$. Now consider a curve in $\{\gamma_n\}$ for $n \geq \max(p, m)$. This will be a timelike curve for f which intersects twice Σ, which will not be an appropriate Cauchy surface.

Let now $\mathcal{M}$ be a manifold geodesically complete and globally hyperbolic with respect to two metrics f and g independently. If there exists a curve in boundary of the manifold such that it can be approached by timelike curves for one of the metrics but for spacelike curves for the companion metric, there is no common Cauchy surface for both metrics.

To prove it, let us consider a family of curves $\{\gamma_n\}$ which converge to the curves γ_g and γ_f in $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_f$ respectively. Suppose that all the curves in $\{\gamma_n\}$ are timelike for g but spacelike for f. If a Cauchy surface Σ exists, it must finish in the boundary. To see why, notice that every timelike curve will intersect Σ once, and thus the series of points of intersection will converge to a point in $\overline{\mathcal{M}}$. This means that Σ must also finish at some point q_f of γ_f. Thus, γ_f will be divided into two regions. Consider now a neighborhood of q_f in $\overline{\mathcal{M}}_f$. Consider two points p_1 and p_2 in the different regions in which Σ divides γ_f. As Σ must be spacelike for g, one of the null cones stemming from p_1 or p_2 will not intersect Σ, which means that Σ is not a good Cauchy surface for g.

[6] As we said before, the map from one of this limit curves to the other one is not necessarily continuous as the topology of $\overline{\mathcal{M}}$ depends on the metric which is used to make the conformal compactification.

Similar (and maybe more general) theorems concerning the lost of global hyperbolicity in quite general bigravity settings can be proved. In this section we just wanted to present the main ideas and leave them for future research.

4 Perturbative Aspects of f–g Bigravity and Massive Gravity

Once we have studied some global issues of bigravity, we can turn back to our original motivation, namely massive gravity. We will divide our study into two possibilities: proportional background metrics (represented by some type II solutions) and not-proportional background metric (represented by type I solutions). Let us begin with the second possibility.

4.1 Type I

To make things simpler, let us consider a type I solutions where both metrics are flat, i.e. $p = q = 0$. From (11) and (12) the metrics can be written as

$$g_{\mu\nu} = \eta_{\mu\nu}, \qquad f_{\mu\nu} = \gamma\tilde{\eta}_{\mu\nu}, \tag{25}$$

where $\gamma = 2/3$ and

$$\tilde{\eta}_{\mu\nu} = \eta_{\mu\nu} - \frac{\gamma}{\beta} P_{\mu\nu}, \tag{26}$$

where we have defined the projector $P_{\mu\nu} = \frac{\beta-1}{\gamma} t_\mu t_\nu$. Besides, $t_\mu = \delta^0_\mu$ and $\eta_{\mu\nu}$ refers to Minkowski metric.

These metrics constitute a solution only for certain vacuum energies [20]

$$\rho_f = -\frac{\zeta}{4}\left(\frac{3}{2}\right)^{4u} \beta^u\{3v + 9\beta(1 - v)\}, \tag{27}$$

$$\rho_g = -\frac{\zeta}{4}\left(\frac{2}{3}\right)^{4v} \beta^{-v}\{3u - 9\beta(1 + u)\}. \tag{28}$$

As both metrics are invariant under a $SO(3)$ transformation, the mass term that we expect to find will be Lorentz-violating, but still $SO(3)$ invariant. However, the kinetic terms will also differ and we can not use the analysis of [24] (see also the appendix of [21]) to disentangle the degrees of freedom. Consider the general perturbation

$$f_{\mu\nu} = \gamma\left(\tilde{\eta}_{\mu\nu} + l_{\mu\nu}\right), \tag{29}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{30}$$

We can project the degrees of freedom of the perturbations using the cosmological spin-decomposition

$$\begin{aligned} h^\alpha_{00} &= 2A^\alpha, \\ h^\alpha_{0i} &= B^\alpha_{|i} + V^\alpha_i, \\ h^\alpha_{ij} &= 2\psi^\alpha\delta_{ij} - 2E^\alpha_{|ij} - 2F^\alpha_{(i|j)} - t^\alpha_{ij}, \end{aligned} \tag{31}$$

where $h^g_{\mu\nu}=h_{\mu\nu}$ and $h^f_{\mu\nu}=l_{\mu\nu}$.

After using the previous decomposition it turns out (see [21]) that the scalar and vector parts are not dynamical, whereas for the tensors and vector we find

$$\frac{1}{2}\Box t^g_{ij}=\kappa_g\zeta\left(\frac{2}{3}\right)^{4v-2}\beta^{-v}(\beta-1)(t^g_{ij}-t^f_{ij}), \tag{32}$$

$$\frac{1}{2}\tilde{\Box}t^f_{ij}=\kappa_f\zeta\left(\frac{3}{2}\right)^{4u+1}\beta^{u}(\beta-1)(t^f_{ij}-t^g_{ij}), \tag{33}$$

where $\tilde{\Box}$ means $\tilde{\eta}^{\mu\nu}\partial_\mu\partial_\nu$. From the previous expressions we see that the tensor modes can be diagonalized to give modes with dispersion relations,

$$\omega^2_\pm=\frac{1}{4\beta}\big(2(\beta+1)k^2+f\pm\sqrt{8\beta k^2(-2k^2+3\beta^{1/2}\kappa C+2C)+(2(\beta+1)k^2+f)^2}\big), \tag{34}$$

where $f=-\beta^{1/2}(3\kappa+2\beta^{1/2})C$, $C=2\kappa_g^2\zeta\beta^{-v}(\beta-1)(\frac{2}{3})^{4v-2}$ and $\kappa=\kappa_g\kappa_f^{-1}$. Studying the propagation of high energy gravitons, we see that each mode propagates in the light cone of each of the metrics. For the low energy expansion, it is easy to see that one of the solutions corresponds to a massive mode while the other one is a massless mode. To have a tachyon-free solution in the tensor sector we must require

$$\beta>0,\qquad C(2\sqrt{\beta}+3\kappa)<0,\qquad (2+3\sqrt{\beta}\kappa)(2\sqrt{\beta}+3\kappa)^{-1}>0. \tag{35}$$

4.2 Type II

Let us now consider as background metrics two proportional de Sitter metrics

$$g_{\mu\nu}=\gamma^{-1}f_{\mu\nu}=\Omega_{\mu\nu},\qquad \Omega_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu=a(\eta)^2(\mathrm{d}\eta^2-\delta_{ij}\mathrm{d}x^i\mathrm{d}x^2), \tag{36}$$

where γ is a solution of (21). Notice that in that case the group of symmetry of both metrics is the same, and thus the mass terms will be covariant. Considering a general perturbation of the type

$$g_{\mu\nu}=\Omega_{\mu\nu}+h_{\mu\nu}, \tag{37}$$

$$f_{\mu\nu}=\gamma(\Omega_{\mu\nu}+l_{\mu\nu}), \tag{38}$$

we arrive at the second order Lagrangian

$$-\frac{1}{2\kappa_+}\sqrt{-g_+}(R_++2\Lambda)-\frac{1}{2\kappa_-}\sqrt{-g_-}(R_-+2\Lambda)+\Omega^{1/2}(M(\bar{l}^\mu_\mu)^2+N(\bar{l}^\nu_\mu\bar{l}^\mu_\nu)), \tag{39}$$

where $\bar{l}_{\mu\nu}=(h-l)_{\mu\nu}$, $\bar{h}_{\mu\nu}=(1+\kappa)^{-1}(h+\kappa l)_{\mu\nu}$ where $\kappa=\gamma\kappa_g\kappa_f^{-1}$, $\kappa_+=\frac{\kappa_f\kappa_g}{\kappa_f+\gamma\kappa_g}$, $\kappa_-=(\kappa_f+\gamma\kappa_g)\gamma^{-1}$, $g_{+\mu\nu}=\Omega_{\mu\nu}+\bar{h}_{\mu\nu}$ and $g_{-\mu\nu}=\Omega_{\mu\nu}+\bar{l}_{\mu\nu}$ and Λ refers to the cosmological constant for Ω, i.e., $\Lambda=\Lambda_g$. Finally the mass terms are given by[7]

$$M=-\frac{\zeta}{4}\gamma^{4v}(6(\gamma^{-1}-1)^2uv+3\gamma^{-1}(\gamma^{-1}-1)(v-u)-\gamma^{-2}), \tag{40}$$

[7] Indeed, for general bigravities proportional metric still constitute a solution with the appropriate γ. In this cases, M and N differ from the ones we find here (see [21]).

$$N = -\frac{\zeta}{4}\gamma^{4v}\gamma^{-1}(3 - 2\gamma^{-1}). \tag{41}$$

From the previous expressions it is clear that one of the gravitons will correspond to a massless perturbation whereas the other one is a general massive perturbation to the de Sitter metric. One may think that the only ghost-free choice would be a Fierz–Pauli type term with $M = -N$. However, as shown in [21], the appearance of the Hubble constant in the kinetic term allows to a wider class of solutions. In particular one finds that for

$$N \leq 0, \qquad 0 \leq M + N \leq \frac{3H^2}{4\kappa_-}, \tag{42}$$

where $a(\eta) = -(H\eta)^{-1}$, there are no-ghosts. Notice that for the Minkowski limit $H \to 0$ the Fierz–Pauli is the only possibility. However, except for the case $M = -N$ (which coincides with the Fierz–Pauli choice) there will be a tachyonic mode for every choice of the parameters. The tachyonic behavior has a cosmological scale of time, i.e. it appears at times

$$-\Delta\eta^2 \gtrsim O(1). \tag{43}$$

As happens in Minkowski space (cf. [24]) we can consider a more general mass term which breaks the $SO(4, 1)$ symmetry to $SO(3)$ (this term may come from type I solutions where one of the metric is de Sitter or from a ghost condensate),

$$L_{\text{int}} = a(\eta)^4(m_0^2\bar{l}_{00}\bar{l}^{00} - 2m_1^2\bar{l}_{0i}\bar{l}^{0i} - m_2^2\bar{l}_{ij}\bar{l}^{ij} + m_3^2\bar{l}_{ii}\bar{l}^{jj} - 2m_4^2\bar{l}_{00}\bar{l}^{ii}). \tag{44}$$

In the appendix of [21] we showed that there are choices of these parameters which lead to ghost and tachyon free Lagrangians different from the Fierz–Pauli one.

5 Offloading the Cosmological Constant

Finally, we will say a few words about an interesting possibility which can occur in bigravity. Remember that throughout the paper we have found some solutions of bigravity with two proportional metrics $f_{\mu\nu} = \gamma g_{\mu\nu}$ where the metrics are solutions of the Einstein's equations with cosmological constants satisfying,

$$\Lambda_g = \gamma\Lambda_f. \tag{45}$$

Even if it is rather difficult to solve the previous equation in general we can study the limits of big or small cosmological constants. The motivation for this is that as we have different scales appearing in the previous equation, we may have a see-saw mechanism which renders one of the cosmological constants tiny and the other one enormous. Thus, the smallness of the observed cosmological constant could be ascribed to an *off-loading* process where one of the metrics absorbs both vacuum energies whereas the other one is practically null. Besides we would like this solution to be ghost-free and that would be better if it has not tachyons.

This problem was addressed in detail in [21] and we found that to get a small cosmological constant for ghost-free solutions, fine tuning is always necessary. This negative result is discouraging but still presents a new possibility for getting a small cosmological constant after some fine tuning is made and without the use of dark energy. Indeed, the role of dark energy would be here played by the matter in the companion world which couples to the coexisting metric.

6 Conclusions

Throughout this paper we have reviewed some results in the global and local aspects of a wide class of bigravity solutions. One of the motivations to do so was to study a non-linear realization for a massive graviton and the possible disappearance of some of the problems that appear at the linear level for massive gravity, such as the vDVZ discontinuity.

The introduction of two Lorentzian metric structures in a manifold raises questions about their compatibility in aspects such as causality and geodesic completeness. To study them we have plotted the light cone structure of one of the metrics into the conformal diagram of the other one. This allows us to see how do the light rays for one of the metric behave with respect to the causal structure of the other one. Even if we restricted to a subclass of spherically symmetric solutions with the same symmetric orbits, we found many interesting phenomena. For instance, the light rays for one of the metrics that move to the future for the other metric at some point can bend and begin to move to the past. This means that by using curves of two interacting globally hyperbolic metrics one may construct closed causal curves. Besides, if one of the metrics can be extended through the boundary of the manifold whereas the other one can not, for the other metric the new region is not visible, as it is always at an infinite proper distance. Even if these problems may appear quite significant, they correspond to the usual possibility of closes causal curves and Cauchy horizons which appears in General Relativity. However, it is quite surprising the fact that in all the known bigravity solutions for the setting that we presented here those closed timelike curves *can not* be constructed. More details on these topics can be found in [20].

We then showed some general results concerning the lost of global hyperbolicity in some settings where both metrics are geodesically complete and globally hyperbolic.

Finally we have computed the spectrum of some of the solutions of the theory and found massive gravitons propagating in these backgrounds. Depending on whether both metrics share the whole symmetry group or just a part of it, the mass spectrum contains terms which violate or not this symmetry. Thus, by studying two flat metric which does not share the whole Lorentz group, but just a $SO(3)$ subgroup, we have found a Lorentz violating mass term. Concerning the covariant masses, we have found general mass terms for massive gravitons in de Sitter backgrounds, which led us to study the ghost and tachyonic free possibilities. In contrast to what happens in Minkowski space, we have found a whole family of mass terms which are ghost free, but only for the Fierz–Pauli case they have no tachyons.

Finally, we have seen that the dynamics of the theory may give rise to the offloading of the cosmological constant of one of the metrics to the other metrics. However we have proven that any ghost free solutions where this happens does nor correspond to natural parameters in the Lagrangian. More details on these topics as long as a study of the general $SO(3)$ invariant mass term in de Sitter will appear in [21].

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